

Lie symmetry analysis of a class of time fractional nonlinear evolution systems

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Abstract

We study a class of nonlinear evolution systems of time fractional partial differential equations depending on an arbitrary function using Lie symmetry analysis. We obtain not only infinitesimal symmetries but also a complete group classification and a classification of group invariant solutions of the systems. The class of systems is divided into two cases depending on an arbitrary function. For each case of the system, the dimensions of Lie algebras generated by infinitesimal symmetries are greater than two, so we calculate one-dimensional optimal systems of the Lie algebras. The reduced systems are also obtained corresponding to the optimal systems. Group invariant solutions are found explicitly for particular cases.

Keywords:

Fractional nonlinear system, Lie symmetry, Optimal system, Invariant solution

1. Introduction

A Lie group analysis becomes an efficient algorithmic approach to study symmetric properties of ordinary and partial differential equations and to solve them [1]-[8]. Recently principal methods of symmetry analysis are extending to solving fractional partial differential equations (FPDE) [9]-[15] and its systems [16]-[18]. The FPDEs have arisen a noticeable interest in mathematics and its applications, such as a study of fractals, acoustics, control theory and signal processing. In this article, we consider the following class of time fractional nonlinear system

$$\begin{cases} \frac{\partial^\alpha u}{\partial t^\alpha} = v, \\ \frac{\partial^\alpha v}{\partial t^\alpha} = b^2(u)u_x, \end{cases} \quad (1)$$

where α is a positive non-integer number and $b(u)$ is a sufficiently differentiable non-constant function. Here the fractional derivative is defined as Riemann-Liouville one

$$\frac{\partial^\alpha u(t, x)}{\partial t^\alpha} = \begin{cases} \frac{\partial^n u}{\partial t^n}, & \alpha = n, n \in \mathbf{N}, \\ \frac{1}{\Gamma(n-\alpha)} \frac{\partial^n}{\partial t^n} \int_0^t \frac{u(x, s)}{(t-s)^{\alpha-n+1}} ds, & n-1 < \alpha < n, n \in \mathbf{N}. \end{cases} \quad (2)$$

In [16] a class of time fractional linear evolution systems

$$\begin{cases} \frac{\partial^\alpha u}{\partial t^\alpha} = C^2(x)v_x, \\ \frac{\partial^\alpha v}{\partial t^\alpha} = u_x, \end{cases} \quad (3)$$

was investigated using the Lie symmetry analysis, where $C(x)$ is a sufficiently differentiable function and α is a positive non-integer number. Also in [17] a nonlinear model of stationary transonic plane-parallel gas flows

$$\begin{cases} \frac{\partial^\alpha u}{\partial t^\alpha} = v, \\ \frac{\partial^\alpha v}{\partial t^\alpha} = -uu_x, \end{cases} \quad \text{where } 0 < \alpha < 1, \quad (4)$$

was studied using the Lie symmetry analysis. The Lie symmetries, some reduced systems of ODEs and some partial solutions of the system (4) was obtained in [17]. By substituting $\bar{u}(x, t) = -u(x, t)$ and $\bar{v}(x, t) = -v(x, t)$ into (4), we get the following equivalent fractional system

$$\begin{cases} \frac{\partial^\alpha \bar{u}}{\partial t^\alpha} = \bar{v}_x, \\ \frac{\partial^\alpha \bar{v}}{\partial t^\alpha} = \bar{u}\bar{u}_x, \end{cases}$$

which coincides to the particular case $b(u) = u^{\frac{1}{2}}$ of the system (1). So the system (1) can be viewed as a nonlinear version of the system (3) and a generalization of the system (4) with respect to the above mentioned substitution, which means the results of [17] are generalized in this paper.

We study the system (1) by the Lie symmetry analysis. More explicitly, we give a complete group classification depending on the function $b(u)$ and describe a structure of Lie algebras generated by infinitesimal symmetries of the system (1). After obtaining the group classification of the system (1), one needs to proceed to finding an optimal system of Lie algebras and the reduced systems of ODEs. Using the optimal systems, we also classify group invariant solutions corresponding to the infinitesimal symmetries for $0 < \alpha < 1$. The organization of the paper is as follows. In Section 2, we cover basics of the Lie symmetry analysis of the system of FPDEs and provide formulas to study the system (1). In Section 3, a complete group classification depending on the function $b(u)$ is carried out. Then in Section 4, we further investigate the structure of Lie algebras of infinitesimal symmetries and determine the optimal systems. Also we reduce the system (1) into the systems of fractional and non-fractional ODEs according to the optimal systems. For particular cases, some explicit solutions are given in Section 5.

2. Lie symmetry analysis for a system of fractional partial differential equations

To begin with, we briefly present basic definitions and necessary formulas regarding the Lie symmetry analysis of a system of FPDEs. The system of time fractional PDEs with two independent variables x and t in a general form is

$$\begin{cases} \frac{\partial^\alpha u(x, t)}{\partial t^\alpha} = F_1(x, t, u, u_x, u_{xx}, \dots, v, v_x, v_{xx}, \dots), \\ \frac{\partial^\alpha v(x, t)}{\partial t^\alpha} = F_2(x, t, u, u_x, u_{xx}, \dots, v, v_x, v_{xx}, \dots), \end{cases} \quad (5)$$

where subscripts denote partial derivatives and α is a positive real number. According to the Lie symmetry analysis, the infinitesimal generator of the system (5) is given by

$$X = \xi \frac{\partial}{\partial x} + \tau \frac{\partial}{\partial t} + \mu \frac{\partial}{\partial u} + \phi \frac{\partial}{\partial v}$$

and its corresponding extended infinitesimal generator is

$$\tilde{X} = X + \mu^{(\alpha)} \frac{\partial}{\partial t^\alpha} + \mu^{(1)} \frac{\partial}{\partial u_x} + \dots + \phi^{(\alpha)} \frac{\partial}{\partial t^\alpha} + \phi^{(1)} \frac{\partial}{\partial v_x} + \dots, \quad (6)$$

where τ, ξ, μ, ϕ are infinitesimals and $\mu^{(\alpha)}, \mu^{(n)}, \phi^{(\alpha)}, \phi^{(n)}$ ($n = 1, 2, \dots$) are extended infinitesimals. The explicit expressions of $\mu^{(n)}$ and $\phi^{(n)}$ are given by

$$\begin{aligned} \mu^{(1)} &= D_x(\mu) - u_x D_x(\xi) - u_t D_x(\tau), \\ \mu^{(2)} &= D_x(\mu^{(1)}) - u_{xx} D_x(\xi) - u_{xt} D_x(\tau), \\ &\vdots \\ \phi^{(1)} &= D_x(\phi) - v_x D_x(\xi) - v_t D_x(\tau), \\ \phi^{(2)} &= D_x(\phi^{(1)}) - v_{xx} D_x(\xi) - v_{xt} D_x(\tau), \\ &\vdots \end{aligned} \quad (7)$$

where D_x is a total derivative operator defined as

$$D_x = \frac{\partial}{\partial x} + u_x \frac{\partial}{\partial u} + u_{xx} \frac{\partial}{\partial u_x} + \cdots + v_x \frac{\partial}{\partial v} + v_{xx} \frac{\partial}{\partial v_x} + \cdots.$$

The α th order extended infinitesimals have the following form [16],[18]

$$\begin{aligned}\mu^{(\alpha)} &= D_t^\alpha(\mu) - \alpha D_t(\tau) \frac{\partial^\alpha u}{\partial t^\alpha} - \sum_{n=1}^{\infty} \binom{\alpha}{n} D_t^n(\xi) D_t^{\alpha-n}(u_x) - \sum_{n=1}^{\infty} \binom{\alpha}{n+1} D_t^{n+1}(\tau) D_t^{\alpha-n}(u), \\ \phi^{(\alpha)} &= D_t^\alpha(\phi) - \alpha D_t(\tau) \frac{\partial^\alpha v}{\partial t^\alpha} - \sum_{n=1}^{\infty} \binom{\alpha}{n} D_t^n(\xi) D_t^{\alpha-n}(v_x) - \sum_{n=1}^{\infty} \binom{\alpha}{n+1} D_t^{n+1}(\tau) D_t^{\alpha-n}(v),\end{aligned}\quad (8)$$

where D_t is a total derivative operator defined as

$$D_t = \frac{\partial}{\partial t} + u_t \frac{\partial}{\partial u} + u_{xt} \frac{\partial}{\partial u_x} + \cdots + v_t \frac{\partial}{\partial v} + v_{xt} \frac{\partial}{\partial v_x} + \cdots$$

and D_t^α is a total fractional derivative operator. Since the lower limit of the integral in (2) is fixed, it should be invariant with respect to point transformations. Then we arrive at an initial condition

$$\tau(x, t, u, v)|_{t=0} = 0. \quad (9)$$

We should note that the last three terms of the right side of each equation of (8) are already in a factored form of the partial derivatives of u and v . The only ones that we need to work on are the first terms $D_t^\alpha(\mu)$, $D_t^\alpha(\phi)$. Let us obtain the explicit, ready to compute form of the extended infinitesimals $\mu^{(\alpha)}$, $\phi^{(\alpha)}$ in the following lemmas.

Lemma 1. *The $\mu^{(\alpha)}$, $\phi^{(\alpha)}$ of the formula (8) can be re-written as*

$$\begin{aligned}\mu^{(\alpha)} &= \frac{\partial^\alpha \mu}{\partial t^\alpha} - u \frac{\partial^\alpha \mu_u}{\partial t^\alpha} - v \frac{\partial^\alpha \mu_v}{\partial t^\alpha} + (\mu_u - \alpha D_t(\tau)) \frac{\partial^\alpha u}{\partial t^\alpha} + \mu_v \frac{\partial^\alpha v}{\partial t^\alpha} - \sum_{n=1}^{\infty} \binom{\alpha}{n} D_t^n(\xi) D_t^{\alpha-n}(u_x) \\ &\quad + \sum_{n=1}^{\infty} \left[\binom{\alpha}{n} \frac{\partial^n \mu_u}{\partial t^n} - \binom{\alpha}{n+1} D_t^{n+1}(\tau) \right] D_t^{\alpha-n}(u) + \sum_{n=1}^{\infty} \binom{\alpha}{n} \frac{\partial^n \mu_v}{\partial t^n} D_t^{\alpha-n}(v) + \mu_1,\end{aligned}\quad (10)$$

$$\begin{aligned}\phi^{(\alpha)} &= \frac{\partial^\alpha \phi}{\partial t^\alpha} - v \frac{\partial^\alpha \phi_v}{\partial t^\alpha} - u \frac{\partial^\alpha \phi_u}{\partial t^\alpha} + (\phi_v - \alpha D_t(\tau)) \frac{\partial^\alpha v}{\partial t^\alpha} + \phi_u \frac{\partial^\alpha u}{\partial t^\alpha} - \sum_{n=1}^{\infty} \binom{\alpha}{n} D_t^n(\xi) D_t^{\alpha-n}(v_x) \\ &\quad + \sum_{n=1}^{\infty} \left[\binom{\alpha}{n} \frac{\partial^n \phi_v}{\partial t^n} - \binom{\alpha}{n+1} D_t^{n+1}(\tau) \right] D_t^{\alpha-n}(v) + \sum_{n=1}^{\infty} \binom{\alpha}{n} \frac{\partial^n \phi_u}{\partial t^n} D_t^{\alpha-n}(u) + \phi_1,\end{aligned}\quad (11)$$

where

$$\begin{aligned}\mu_1 &= \sum_{n=2}^{\infty} \sum_{m_1+m_2=2}^n \sum_{\substack{k_1=0,\dots,m_1 \\ k_2=0,\dots,m_2 \\ k_1+k_2 \geq 2}} \sum_{r_1=0}^{k_1} \sum_{r_2=0}^{k_2} \binom{\alpha}{n} \binom{n}{m_1} \binom{n-m_1}{m_2} \binom{k_1}{r_1} \binom{k_2}{r_2} \frac{1}{k_1!k_2!} \frac{t^{n-\alpha}}{\Gamma(n+1-\alpha)} \\ &\quad \times (-u)^{r_1} (-v)^{r_2} \frac{\partial^{m_1} u^{k_1-r_1}}{\partial t^{m_1}} \frac{\partial^{m_2} v^{k_2-r_2}}{\partial t^{m_2}} \frac{\partial^{n-m_1-m_2+k_1+k_2} \mu}{\partial t^{n-m_1-m_2} \partial u^{k_1} \partial v^{k_2}},\end{aligned}$$

and

$$\begin{aligned}\phi_1 &= \sum_{n=2}^{\infty} \sum_{m_1+m_2=2}^n \sum_{\substack{k_1=0,\dots,m_1 \\ k_2=0,\dots,m_2 \\ k_1+k_2 \geq 2}} \sum_{r_1=0}^{k_1} \sum_{r_2=0}^{k_2} \binom{\alpha}{n} \binom{n}{m_1} \binom{n-m_1}{m_2} \binom{k_1}{r_1} \binom{k_2}{r_2} \frac{1}{k_1!k_2!} \frac{t^{n-\alpha}}{\Gamma(n+1-\alpha)} \\ &\quad \times (-u)^{r_1} (-v)^{r_2} \frac{\partial^{m_1} u^{k_1-r_1}}{\partial t^{m_1}} \frac{\partial^{m_2} v^{k_2-r_2}}{\partial t^{m_2}} \frac{\partial^{n-m_1-m_2+k_1+k_2} \phi}{\partial t^{n-m_1-m_2} \partial u^{k_1} \partial v^{k_2}}.\end{aligned}$$

PROOF. It is sufficient to prove the formula of $\mu^{(\alpha)}$. It can be proved by an expansion of the first term $D_t^\alpha(\mu)$ of (8) applying a generalized Leibniz rule and a generalized chain rule [19],[20]:

$$\begin{aligned}
D_t^\alpha(\mu) &= \sum_{n=0}^{\infty} \binom{\alpha}{n} \frac{t^{n-\alpha}}{\Gamma(n+1-\alpha)} D_t^n(\mu) \\
&= \sum_{n=0}^{\infty} \sum_{m_1+m_2=0}^n \binom{\alpha}{n} \frac{t^{n-\alpha}}{\Gamma(n+1-\alpha)} \binom{n}{m_1} \binom{n-m_1}{m_2} \left[\frac{\partial^{n-m_1-m_2} \partial^{m_1} \partial^{m_2} \mu(x, t, u(x, t_1), v(x, t_2))}{\partial t^{n-m_1-m_2} \partial t_1^{m_1} \partial t_2^{m_2}} \right] \Big|_{\substack{t_1=t \\ t_2=t}} \\
&= \sum_{n=0}^{\infty} \sum_{m_1+m_2=0}^n \sum_{k_1=0}^{m_1} \sum_{r_1=0}^{k_1} \sum_{k_2=0}^{m_2} \sum_{r_2=0}^{k_2} \binom{\alpha}{n} \frac{t^{n-\alpha}}{\Gamma(n+1-\alpha)} \binom{n}{m_1} \binom{n-m_1}{m_2} \binom{k_1}{r_1} \binom{k_2}{r_2} \frac{1}{k_1! k_2!} \\
&\quad \times (-u)^{r_1} (-v)^{r_2} \frac{\partial^{m_1} u^{k_1-r_1}}{\partial t^{m_1}} \frac{\partial^{m_2} v^{k_2-r_2}}{\partial t^{m_2}} \frac{\partial^{n-m_1-m_2+k_1+k_2} \mu}{\partial t^{n-m_1-m_2} \partial u^{k_1} \partial v^{k_2}}.
\end{aligned} \tag{12}$$

Since μ_1 equals to the partial sum of the summation (12) when the sum of k_1 and k_2 is greater than 1, we need to take a closer look at only the case of $k_1 + k_2 \leq 1$. All possible summations for values of $(m_1, m_2, k_1, k_2, r_1, r_2)$ satisfying the inequality is divided into five cases, which are indexed by i . We denote the resulting sum by $D_t^\alpha(\mu)_i$ and these are shown in the following table.

Subcase i	$(m_1, m_2, k_1, k_2, r_1, r_2)$	$D_t^\alpha(\mu)_i$
1)	$(0, 0, 0, 0, 0, 0)$	$D_t^\alpha(\mu)_1 = \sum_{n=0}^{\infty} \binom{\alpha}{n} D_t^{\alpha-n} 1 \cdot \frac{\partial^n \mu}{\partial t^n} = \frac{\partial^\alpha \mu}{\partial t^\alpha}$
2)	$(0, 0, 1, 0, 1, 0)$	$D_t^\alpha(\mu)_2 = \sum_{n=0}^{\infty} \binom{\alpha}{n} D_t^{\alpha-n} 1 (-u) \frac{\partial^{n+1} \mu}{\partial t^n \partial u} = -u \frac{\partial^\alpha \mu_u}{\partial t^\alpha}$
3)	$(0, 0, 0, 1, 0, 1)$	$D_t^\alpha(\mu)_3 = -v \frac{\partial^\alpha \mu_v}{\partial t^\alpha}$
4)	$(m_1, 0, 1, 0, 0, 0)$	$D_t^\alpha(\mu)_4 = \mu_u \frac{\partial^\alpha \mu}{\partial t^\alpha} + \sum_{n=1}^{\infty} \binom{\alpha}{n} \frac{\partial^n \mu_u}{\partial t^n} D_t^{\alpha-n} u$
5)	$(0, m_2, 0, 1, 0, 0)$	$D_t^\alpha(\mu)_5 = \mu_v \frac{\partial^\alpha \mu}{\partial t^\alpha} + \sum_{n=1}^{\infty} \binom{\alpha}{n} \frac{\partial^n \mu_v}{\partial t^n} D_t^{\alpha-n} v$

By summing up the above five cases $D_t^\alpha(\mu)_i$ ($i = 1, \dots, 5$) and μ_1 , then substituting it into (8) one can obtain the explicit form of $\mu^{(\alpha)}$ as in the lemma. The formula of $\varphi^{(\alpha)}$ can be obtained in similar steps. \square

Note: If $\mu^{(\alpha)}$ or $\phi^{(\alpha)}$ is linear on u and v then $\mu_1 = 0$ and $\phi_1 = 0$.

The infinitesimal invariance criterion for the system (5) according to the Lie symmetry analysis is

$$\begin{cases} \tilde{X}(u_{t^\alpha} - F_1(x, t, u, u_x, u_{xx}, \dots, v, v_x, v_{xx}, \dots)) \Big|_{(5)} = 0, \\ \tilde{X}(v_{t^\alpha} - F_2(x, t, u, u_x, u_{xx}, \dots, v, v_x, v_{xx}, \dots)) \Big|_{(5)} = 0, \end{cases} \tag{14}$$

where \tilde{X} is given by (6),(7),(10) and (11). Now we are ready to investigate the infinitesimal symmetries of the system (1) following the above mentioned Lie symmetry analysis of the time fractional system.

3. Lie symmetry analysis of the fractional nonlinear evolution system (1)

In this section we study the system (1) using the formulas of the previous section. Since the system (1) admits different symmetry groups according to function $b(u)$, the system (1) is divided into cases depending on the function $b(u)$. For each cases we obtain the infinitesimal symmetries.

According to (14) we have the following invariance criterion for the system (1)

$$\begin{cases} \tilde{X}(u_{t^\alpha} - v_x)|_{(1)} = 0, \\ \tilde{X}(v_{t^\alpha} - b^2(u)u_x)|_{(1)} = 0, \end{cases}$$

which is in explicit form

$$\begin{cases} (\mu^{(\alpha)} - \phi^{(1)})|_{(1)} = 0, \\ (\phi^{(\alpha)} - 2\mu b b' u_x - b^2 \mu^{(1)})|_{(1)} = 0. \end{cases} \quad (15)$$

From the system (15) we get the following overdetermined system of determining equations by equating the coefficients of linearly independent partial derivatives $D_t^{\alpha-n}u$, $D_t^{\alpha-n}v$, $D_t^{\alpha-n}u_x$, $D_t^{\alpha-n}v_x$, v_x , u_x , v_t , $u_x v_t$, $v_x v_t$, $u_x v_x$ and v_x^2 to zero:

$$\begin{aligned} \binom{\alpha}{n} \frac{\partial^n \mu_u}{\partial t^n} - \binom{\alpha}{n+1} D_t^{n+1} \tau &= 0, \quad n = 1, 2, \dots, \\ \frac{\partial^n \mu_v}{\partial t^n} &= 0, \quad n = 1, 2, \dots, \\ D_t^n(\xi) &= 0, \quad n = 1, 2, \dots, \\ \mu_u - \alpha D_t(\tau) - \phi_v + \xi_x &= 0, \\ b^2 \mu_v - \phi_u &= 0, \\ \frac{\partial^\alpha \mu}{\partial t^\alpha} - v \frac{\partial^\alpha \mu_v}{\partial t^\alpha} - u \frac{\partial^\alpha \mu_u}{\partial t^\alpha} - \phi_x + \mu_1 &= 0, \\ \frac{\partial^n \phi_u}{\partial t^n} &= 0, \quad n = 1, 2, \dots, \\ \binom{\alpha}{n} \frac{\partial^n \phi_v}{\partial t^n} - \binom{\alpha}{n+1} D_t^{n+1} \tau &= 0, \quad n = 1, 2, \dots, \\ b^2 \phi_v - \alpha b^2 D_t(\tau) - 2b b' \mu - b^2 \mu_u + b^2 \xi_x &= 0, \\ \frac{\partial^\alpha \phi}{\partial t^\alpha} - v \frac{\partial^\alpha \phi_v}{\partial t^\alpha} - u \frac{\partial^\alpha \phi_u}{\partial t^\alpha} - b^2 \mu_x + \phi_1 &= 0, \\ \tau_x = \tau_u = \tau_v &= 0, \\ \xi_u = \xi_v &= 0. \end{aligned}$$

Analyzing the above overdetermined system with the initial condition (9) we get infinitesimal symmetries for the following two cases depending on the function $b(u)$.

Case 1. For an arbitrary non-constant function $b(u)$, the infinitesimals are found to be

$$\tau = \frac{s_1}{\alpha} t, \quad \xi = s_1 x + s_2, \quad \mu = 0, \quad \phi = s(t),$$

here s_1, s_2 are arbitrary constants and $s(t)$ is a solution of the equation $\frac{d^\alpha s(t)}{dt^\alpha} = 0$. Hence the infinitesimal symmetries are

$$X_1 = \frac{\partial}{\partial x}, \quad X_2 = x \frac{\partial}{\partial x} + \frac{t}{\alpha} \frac{\partial}{\partial t}, \quad X_3 = s(t) \frac{\partial}{\partial v}.$$

Case 2. For $b(u) = ku^m$, $k, m \neq 0$, we have an additional infinitesimal symmetry. So the infinitesimal symmetries are

$$X_1, \quad X_2, \quad X_3, \quad X_4 = x \frac{\partial}{\partial x} + \frac{u}{m} \frac{\partial}{\partial u} + \frac{1+m}{m} v \frac{\partial}{\partial v}.$$

Since we have a complete group classification of the system (1) depending on the function $b(u)$, now we can think of one-dimensional optimal systems of Lie algebras of infinitesimal symmetries and the classification of group invariant solutions. Before moving to the next section we should note that the solution of the equation $\frac{d^\alpha s(t)}{dt^\alpha} = 0$ is

$$s(t) = c_1 t^{\alpha-1} + c_2 t^{\alpha-2} + \dots + c_n t^{\alpha-n},$$

where c_i are arbitrary constants and n is a positive integer such that $n - 1 < \alpha < n$.

4. Optimal systems and classifications of invariant solutions

In this section we classify the group invariant solutions of the system (1) corresponding to infinitesimal symmetries for the case of $0 < \alpha < 1$. The definition of invariant solutions is given as following [16].

Definition 1. $(u(x, t), v(x, t))$ is said to be an invariant solution of the system (5) corresponding to the infinitesimal symmetry $X = \xi \frac{\partial}{\partial x} + \tau \frac{\partial}{\partial t} + \mu \frac{\partial}{\partial u} + \phi \frac{\partial}{\partial v}$ if and only if

- i) $(u(x, t), v(x, t))$ satisfies the system (5),
- ii) $(u(x, t), v(x, t))$ satisfies the invariant surface condition of X , which is

$$\begin{cases} \xi \frac{\partial u}{\partial x} + \tau \frac{\partial u}{\partial t} - \mu = 0, \\ \xi \frac{\partial v}{\partial x} + \tau \frac{\partial v}{\partial t} - \phi = 0. \end{cases}$$

The invariant solutions of the system (1) corresponding to any infinitesimal symmetry can be obtained by Lie symmetry transformations on the invariant solutions corresponding to the infinitesimal symmetries of any optimal system of one-dimensional subalgebras of infinitesimal symmetries [7]. Therefore all we have to do is to describe the invariant solutions corresponding to infinitesimal symmetries of the optimal system. More explicitly we express the invariant solutions as solutions to reduced systems of ODEs. To obtain the optimal system we construct the commutator and adjoint tables of Lie algebras of infinitesimal symmetries of the system (1). The commutator table is given by the Lie bracket operation $[X_i, X_j] = X_i(X_j) - X_j(X_i)$, where $i, j = 1, 2, 3$ for Case 1 and $i, j = 1, \dots, 4$ for Case 2. The adjoint table is given by

$$Ad(e^{\varepsilon X_i})X_j = X_j - \varepsilon[X_i, X_j] + \frac{\varepsilon^2}{2}[X_i, [X_i, X_j]] - \dots,$$

where $\varepsilon \in \mathbb{R}$. The optimal systems are found following the procedure of Chapter 10.2 of [8]. For each case of the function $b(u)$ of the system (1) the optimal systems and its corresponding reduced systems are given in the subsequent subsections.

4.1. Case 1.

We know from the previous section that for an arbitrary non-constant function $b(u)$ and $0 < \alpha < 1$, the system (1) has the following infinitesimal symmetries

$$X_1 = \frac{\partial}{\partial x}, \quad X_2 = x \frac{\partial}{\partial x} + t \frac{\partial}{\partial t}, \quad X_3 = t^{\alpha-1} \frac{\partial}{\partial v}.$$

The commutator and adjoint tables of the Lie algebras generated by infinitesimal symmetries are given below with the index i denoting row and j denoting column.

$[X_i, X_j]$	X_1	X_2	X_3
X_1	0	X_1	0
X_2	$-X_1$	0	$\frac{\alpha-1}{\alpha}X_3$
X_3	0	$-\frac{\alpha-1}{\alpha}X_3$	0

Table 1: Commutator table of Case 1.

$Ad(e^{\varepsilon X_i})X_j$	X_1	X_2	X_3
X_1	X_1	$X_2 - \varepsilon X_1$	X_3
X_2	$e^\varepsilon X_1$	X_2	$e^{-\frac{\alpha-1}{\alpha}\varepsilon} X_3$
X_3	X_1	$X_2 + \frac{\alpha-1}{\alpha}\varepsilon X_3$	X_3

Table 2: Adjoint table of Case 1.

Therefore the one-dimensional optimal system of the Lie algebra generated by X_1 , X_2 and X_3 is

$$U_1 = X_1 + aX_3, \quad U_2 = X_2, \quad U_3 = X_3, \quad a \in \mathbb{R}.$$

Using the standard characteristic method we obtain the invariant solutions and reduced systems of ODEs of the system (1) corresponding to each symmetry U_j , which are shown in Table 3.

U_j	Invariant solutions $(u_j(x, t), v_j(x, t))$	Reduced systems of ODE _j
U_1	$\begin{cases} u(x, t) = \varphi(t), \\ v(x, t) = \psi(t) + at^{\alpha-1}x, \end{cases}$	$\begin{cases} \frac{d^\alpha \varphi}{dt^\alpha} = at^{\alpha-1}, \\ \frac{d^\alpha \psi}{dt^\alpha} = 0, \end{cases} \quad a \in \mathbb{R}$
U_2	$\begin{cases} u(x, t) = \varphi(z), \\ v(x, t) = \psi(z), \end{cases} \quad \text{with } z = tx^{-\frac{1}{\alpha}}$	$\begin{cases} \frac{d^\alpha \varphi}{dz^\alpha} = -\frac{1}{\alpha}\psi', \\ \frac{d^\alpha \psi}{dz^\alpha} = -\frac{1}{\alpha}b^2(\varphi)\varphi', \end{cases}$
U_3	There are no invariant solutions.	

Table 3: The optimal systems and reduced systems of the system (1) for an arbitrary non-constant $b(u)$

We will encounter the reduced systems of ODEs corresponding to U_1 in later sections and the reduced system of ODEs corresponding to U_2 depends on an arbitrary function $b(u)$, so we refrain from solving it here.

4.2. Case 2.

For $b(u) = ku^m$, k and m are non-zero real numbers and $0 < \alpha < 1$, the system (1) becomes

$$\begin{cases} \frac{\partial^\alpha u}{\partial t^\alpha} = v_x \\ \frac{\partial^\alpha v}{\partial t^\alpha} = k^2 u^{2m} u_x \end{cases} \quad (16)$$

and the corresponding infinitesimal symmetries are

$$X_1 = \frac{\partial}{\partial x}, \quad X_2 = x \frac{\partial}{\partial x} + \frac{t}{\alpha} \frac{\partial}{\partial t}, \quad X_3 = t^{\alpha-1} \frac{\partial}{\partial v}, \quad X_4 = x \frac{\partial}{\partial x} + \frac{u}{m} \frac{\partial}{\partial u} + \frac{m+1}{m} v \frac{\partial}{\partial v}.$$

The optimal systems of a Lie algebra depend on the structure of the Lie algebra. Since the structure of the Lie algebra generated by the above X_1 , X_2 , X_3 , and X_4 depends on parameters m and α , we study the system (16) for the following two subcases $2m\alpha + \alpha - m \neq 0$ and $2m\alpha + \alpha - m = 0$.

Case 2.1.

Let $2m\alpha + \alpha - m \neq 0$. For the Lie algebra generated by the symmetries X_i ($i = 1, \dots, 4$) we choose a new basis Y_i ($i = 1, \dots, 4$) such that the Lie algebra consists of the direct sum of two subalgebras L_1 and L_2 , where L_1 is generated by Y_1 , Y_2 and L_2 is generated by Y_3 , Y_4 . The chosen new basis is

$$\begin{aligned} Y_1 &= X_1, & Y_2 &= \frac{(m+1)\alpha}{2m\alpha + \alpha - m} X_2 + \frac{m(\alpha-1)}{2m\alpha + \alpha - m} X_4, \\ Y_3 &= X_3, & Y_4 &= \frac{m\alpha}{2m\alpha + \alpha - m} (X_4 - X_2). \end{aligned}$$

The commutator and adjoint tables of Y_i ($i = 1, \dots, 4$) are given below with the index i denoting row and j denoting column.

$[Y_i, Y_j]$	Y_1	Y_2	Y_3	Y_4
Y_1	0	Y_1	0	0
Y_2	$-Y_1$	0	0	0
Y_3	0	0	0	Y_3
Y_4	0	0	$-Y_3$	0

Table 4: Commutator table of Case 2.1.

$Ad(e^{\varepsilon Y_i})Y_j$	Y_1	Y_2	Y_3	Y_4
Y_1	Y_1	$Y_2 - \varepsilon Y_1$	Y_3	Y_4
Y_2	$e^\varepsilon Y_1$	Y_2	Y_3	Y_4
Y_3	Y_1	Y_2	Y_3	$Y_4 - \varepsilon Y_3$
Y_4	Y_1	Y_2	$e^\varepsilon Y_3$	Y_4

Table 5: Adjoint table of Case 2.1.

The optimal system in the new and original bases is

$$\begin{aligned}
U_1 &= Y_1 + aY_3 = X_1 + aX_3, \quad a = 0, 1, -1, \\
U_2 &= Y_1 + \frac{(2m\alpha + \alpha - m)a}{m\alpha} Y_4 \\
&= X_1 - aX_2 + aX_4, \quad a = 1, -1, \\
U_3 &= (2m\alpha + \alpha - m)Y_2 + aY_3 \\
&= (m+1)\alpha X_2 + aX_3 + m(\alpha-1)X_4, \quad a = 1, -1, \\
U_4 &= Y_2 + \frac{(2m\alpha + \alpha - m)a - m\alpha + m}{m\alpha} Y_4 \\
&= (1-a)X_2 + aX_4, \quad a \in \mathbb{R}, \\
U_5 &= \frac{2m\alpha + \alpha - m}{m\alpha} Y_4 = -X_2 + X_4, \\
U_6 &= Y_3 = X_3.
\end{aligned}$$

In Table 6 and Table 7, we show the similarity variables z_j , invariant solutions $(u_j(x, t), v_j(x, t))$ expressed as solutions $(\varphi(z), \psi(z))$ of the reduced systems of ODEs and the reduced systems of ODEs corresponding to U_j in the optimal system.

Note that due to the divergence of the integral of the definition (2) of Riemann-Liouville derivative, $\frac{d^\alpha}{dt^\alpha}(t^p)$ is not defined for $p \leq -1$ [19]. So we need an additional assumption, which appears in Table 6, on invariant solutions corresponding to the symmetry U_5 .

U_j	z_j	Invariant solutions $(u_j(x, t), v_j(x, t))$
U_1	t	$\begin{cases} u(x, t) = \varphi(t), \\ v(x, t) = \psi(t) + ax t^{\alpha-1}, \end{cases} \quad a = 0, 1, -1$
U_2	$t \exp\left(\frac{a}{\alpha}x\right)$	$\begin{cases} u(x, t) = \exp\left(\frac{a}{m}x\right) \varphi(z), \\ v(x, t) = a \exp\left(\frac{(m+1)a}{m}x\right) \psi(z), \end{cases} \quad a = 1, -1$
U_3	$tx^{-\frac{m+1}{2m\alpha+\alpha-m}}$	$\begin{cases} u(x, t) = x^{\frac{\alpha-1}{2m\alpha+\alpha-m}} \varphi(z), \\ v(x, t) = x^{\frac{(m+1)(\alpha-1)}{2m\alpha+\alpha-m}} \psi(z) + \frac{a}{2m\alpha+\alpha-m} t^{\alpha-1} \ln(x), \end{cases} \quad a = 1, -1$
U_4	$tx^{\frac{a-1}{\alpha}}$	$\begin{cases} u(x, t) = x^{\frac{a}{m}} \varphi(z), \\ v(x, t) = x^{\frac{(m+1)a}{m}} \psi(z), \end{cases} \quad a \in \mathbb{R}$
U_5	x	$\begin{cases} u(x, t) = t^{-\frac{\alpha}{m}} \varphi(x), \\ v(x, t) = t^{-\frac{(m+1)\alpha}{m}} \psi(x), \end{cases} \quad m < 0 \text{ or } m > \frac{\alpha}{1-\alpha}$
U_6		No invariant solutions

Table 6: Similarity variables z_j and invariant solutions (u_j, v_j) of Case 2.1.

U_j	Reduced system of ODE _j
U_1	$\begin{cases} \frac{d^\alpha \varphi(t)}{dt^\alpha} = at^{\alpha-1}, \\ \frac{d^\alpha \psi(t)}{dt^\alpha} = 0, \end{cases} \quad a = 0, 1, -1$
U_2	$\begin{cases} \frac{d^\alpha \varphi}{dz^\alpha} = a^2 \left(\frac{m+1}{m} \psi + \frac{1}{\alpha} z \psi' \right), \\ \frac{d^\alpha \psi}{dz^\alpha} = k^2 \varphi^{2m} \left(\frac{1}{m} \varphi + \frac{1}{\alpha} z \varphi' \right), \end{cases} \quad a = 1, -1$
U_3	$\begin{cases} \frac{d^\alpha \varphi}{dz^\alpha} = \frac{m+1}{2m\alpha+\alpha-m} ((\alpha-1)\psi - z\psi') + \frac{a}{2m\alpha+\alpha-m} z^{\alpha-1}, \\ \frac{d^\alpha \psi}{dz^\alpha} = \frac{k^2}{2m\alpha+\alpha-m} \varphi^{2m} ((\alpha-1)\varphi - (m+1)z\varphi'), \end{cases} \quad a = 1, -1$
U_4	$\begin{cases} \frac{d^\alpha \varphi}{dz^\alpha} = \frac{(m+1)a}{m} \psi + \frac{a-1}{\alpha} z \psi', \\ \frac{d^\alpha \psi}{dz^\alpha} = k^2 \varphi^{2m} \left(\frac{a}{m} \varphi + \frac{a-1}{\alpha} z \varphi' \right), \end{cases} \quad a \in \mathbb{R}$
U_5	$\begin{cases} \psi'(x) = \frac{\Gamma(1-\frac{\alpha}{m})}{\Gamma(1-\frac{(m+1)\alpha}{m})} \varphi(x), \\ k^2 \varphi^{2m} \varphi'(x) = \frac{\Gamma(1-\frac{(m+1)\alpha}{m})}{\Gamma(1-\frac{(2m+1)\alpha}{m})} \psi(x), \end{cases} \quad m < 0 \text{ or } m > \frac{\alpha}{1-\alpha}$

Table 7: Reduced systems of ODEs of Case 2.1.

Case 2.2.

Let $2m\alpha + \alpha - m = 0$. So we obtain $m = \frac{\alpha}{1-2\alpha}$ and $\alpha \neq \frac{1}{2}$. For the Lie algebra generated by the symmetries X_i ($i = 1, \dots, 4$) we choose a new basis Y_i ($i = 1, \dots, 4$) such that the Lie algebra consists of the direct sum of two subalgebras L_1 and L_2 , where L_1 is generated by Y_1, Y_2, Y_3 and L_2 is generated by Y_4 . The new basis is

$$Y_1 = X_1, \quad Y_2 = X_4, \quad Y_3 = X_3, \quad Y_4 = X_4 - X_2.$$

The commutator and adjoint tables of Y_i ($i = 1, \dots, 4$) are given below with the index i denoting row and j denoting column.

$[Y_i, Y_j]$	Y_1	Y_2	Y_3	Y_4
Y_1	0	Y_1	0	0
Y_2	$-Y_1$	0	$-\frac{1-\alpha}{\alpha}Y_3$	0
Y_3	0	$\frac{1-\alpha}{\alpha}Y_3$	0	0
Y_4	0	0	0	0

Table 8: Commutator table of Case 2.2.

$Ad(e^{\varepsilon Y_i})Y_j$	Y_1	Y_2	Y_3	Y_4
Y_1	Y_1	$Y_2 - \varepsilon Y_1$	Y_3	Y_4
Y_2	$e^\varepsilon Y_1$	Y_2	$e^{\frac{1-\alpha}{\alpha}\varepsilon}Y_3$	Y_4
Y_3	Y_1	$Y_2 - \frac{1-\alpha}{\alpha}\varepsilon Y_3$	Y_3	Y_4
Y_4	Y_1	Y_2	Y_3	Y_4

Table 9: Adjoint table of Case 2.2.

The optimal system in the new and original bases is

$$\begin{aligned}
U_1 &= Y_1 + aY_3 = X_1 + aX_3, \quad a \in \mathbb{R}, \\
U_2 &= Y_1 + a_1Y_3 + a_2Y_4 \\
&= X_1 - a_2X_2 + a_1X_3 + a_2X_4, \quad (a_1, a_2) \in \{(a, \pm 1) | a \in \mathbb{R}\}, \\
U_3 &= Y_2 + (a-1)Y_4 = (1-a)X_2 + aX_4, \quad a \in \mathbb{R}, \\
U_4 &= aY_3 + Y_4 = -X_2 + aX_3 + X_4, \quad a = 0, 1, -1, \\
U_5 &= Y_3 = X_3.
\end{aligned}$$

In the following Table 10, we show the similarity variables z_j and invariant solutions $(u_j(x, t), v_j(x, t))$, which are expressed as solutions to reduced systems. Then, in Table 11, we give the reduced systems of ODEs corresponding to the above optimal system.

U_j	z_j	Invariant solutions $(u_j(x, t), v_j(x, t))$
U_1	t	$\begin{cases} u(x, t) = \varphi(t), \\ v(x, t) = \psi(t) + axt^{\alpha-1}, \end{cases} \quad a \in \mathbb{R},$
U_2	$t \exp(\frac{a_2}{\alpha}x)$	$\begin{cases} u(x, t) = \exp\left(\frac{a_2(1-2\alpha)}{\alpha}x\right) \varphi(z), \\ v(x, t) = a_2 \exp\left(\frac{a_2(1-\alpha)}{\alpha}x\right) \psi(z) + a_1xt^{\alpha-1}, \end{cases} \quad (a_1, a_2) \in \{(a, \pm 1) a \in \mathbb{R}\}$
U_3	$z = tx^{\frac{\alpha-1}{\alpha}}$	$\begin{cases} u(x, t) = x^{\frac{a(1-2\alpha)}{\alpha}} \varphi(z), \\ v(x, t) = x^{\frac{a(1-\alpha)}{\alpha}} \psi(z), \end{cases} \quad a \in \mathbb{R}$
U_4	x	$\begin{cases} u(x, t) = t^{2\alpha-1} \varphi(x), \\ v(x, t) = t^{\alpha-1} \psi(x) - a\alpha t^{\alpha-1} \ln(t), \end{cases} \quad a = 0, 1, -1$
U_5		No invariant solutions

Table 10: Similarity variables z_j and invariant solutions (u_j, v_j) of Case 2.2.

U_j	Reduced system of ODE _j
U_1	$\begin{cases} \frac{d^\alpha \varphi(t)}{dt^\alpha} = at^{\alpha-1}, \\ \frac{d^\alpha \psi(t)}{dt^\alpha} = 0, \end{cases} \quad a \in \mathbb{R}$
U_2	$\begin{cases} \frac{d^\alpha \varphi}{dz^\alpha} = \frac{a_2}{\alpha} ((1-\alpha)\psi + z\psi') + a_1 z^{\alpha-1}, \\ \frac{d^\alpha \psi}{dz^\alpha} = \frac{k^2}{\alpha} \varphi^{\frac{2\alpha}{1-2\alpha}} ((1-2\alpha)\varphi + z\varphi'), \end{cases} \quad (a_1, a_2) \in \{(a, \pm 1) a \in \mathbb{R}\}$
U_3	$\begin{cases} \frac{d^\alpha \varphi}{dz^\alpha} = \frac{1}{\alpha} (a(1-\alpha)\psi + (a-1)z\psi'), \\ \frac{d^\alpha \psi}{dz^\alpha} = \frac{k^2}{\alpha} \varphi^{\frac{2\alpha}{1-2\alpha}} (a(1-2\alpha)\varphi + (a-1)z\varphi'), \end{cases} \quad a \in \mathbb{R}$
U_4	$\begin{cases} \psi'(x) = \frac{\Gamma(2\alpha)}{\Gamma(\alpha)} \varphi(x), \\ k^2 \varphi^{\frac{2\alpha}{1-2\alpha}} \varphi'(x) = -a\Gamma(\alpha+1), \end{cases} \quad a = 0, 1, -1,$

Table 11: Reduced systems of ODEs of Case 2.2.

From seeing the elements of the optimal systems of Case 2.1 and Case 2.2 we can make the following three conclusions for the case of $m = \frac{\alpha}{1-2\alpha}$.

1. Two invariant solutions corresponding to U_1 of Case 2.1 and Case 2.2 coincide.
2. The invariant solutions corresponding to U_3 of Case 2.1 coincide to the invariant solutions corresponding to U_4 of Case 2.2.
3. Also the invariant solutions corresponding to U_4 of Case 2.1 coincide to the invariant solutions corresponding to U_3 of Case 2.2.

Even though the elements of optimal systems of Case 2.1 and Case 2.2 are essentially correspond to each other except the element U_2 , the reduced systems of ODEs of Case 2.1 and Case 2.2 are different depending on the choice of similarity variable z .

In the next section we will give some explicit invariant solutions to the fractional system (1) by solving the obtained reduced systems of ODEs in this section.

5. Explicit solutions to the fractional nonlinear evolution system (1)

In a general case, solving fractional order nonlinear systems of ODEs is a challenging problem, but in this section we give a several explicit solutions to the reduced systems of ODEs that are obtained in the previous section. Then using the solutions of reduced systems of ODEs, we obtain some group invariant solutions of the system (1).

5.1. Reduced system of ODE₁

The reduced systems of ODE₁ of all three cases are essentially same, the solution of which is

$$\begin{cases} \varphi(t) = \frac{a}{\Gamma(\alpha)} t^{2\alpha-1} + c_1 t^{\alpha-1}, \\ \psi(t) = c_2 t^{\alpha-1}, \end{cases}$$

where, c_1, c_2 are arbitrary constants. From here the invariant solutions of the system (1) becomes

$$\begin{cases} u(x, t) = \frac{a}{\Gamma(\alpha)} t^{2\alpha-1} + c_1 t^{\alpha-1}, \\ v(x, t) = (c_2 + ax) t^{\alpha-1}. \end{cases} \quad (17)$$

5.2. Reduced system of ODE₄ of Case 2.1

The reduced system of ODE₄ of Case 2.1 has the following general form

$$\begin{cases} \frac{d^\alpha \varphi}{dz^\alpha} = a_1 \psi + a_2 z \psi_z, \\ \frac{d^\alpha \psi}{dz^\alpha} = \varphi^{2m} (b_1 \varphi + b_2 z \varphi_z), \end{cases} \quad (18)$$

where a_1, a_2, b_1 and b_2 are constants. We will formulate the following lemma on a solution of the system (18).

Lemma 2. Let us assume that parameter m satisfies $m < 0$ or $m > \frac{\alpha}{1-\alpha}$. If the following inequalities hold

$$m \neq \frac{\alpha}{1-2\alpha}, \quad a_1 - \frac{m+1}{m}a_2\alpha \neq 0, \quad b_1 - \frac{1}{m}b_2\alpha \neq 0,$$

then the system (18) has a solution of the form $\varphi(z) = c_1 z^{\lambda_1}$, $\psi(z) = c_2 z^{\lambda_2}$, where

$$\begin{aligned} \lambda_1 &= -\frac{\alpha}{m}, & \lambda_2 &= -\frac{(m+1)\alpha}{m}, \\ c_1 &= \left(\frac{\Gamma(1 - \frac{\alpha}{m}) m^2}{\Gamma(1 - \frac{(2m+1)\alpha}{m}) (ma_1 - (m+1)a_2\alpha)} \frac{1}{(mb_1 - b_2\alpha)} \right)^{\frac{1}{2m}}, \\ c_2 &= \frac{1}{\Gamma(1 - \frac{(m+1)\alpha}{m})} \left(\frac{\Gamma(1 - \frac{\alpha}{m})^{2m+1} m^{2m+2}}{\Gamma(1 - \frac{(2m+1)\alpha}{m})} \frac{1}{(ma_1 - (m+1)a_2\alpha)^{2m+1} (mb_1 - b_2\alpha)} \right)^{\frac{1}{2m}}. \end{aligned}$$

PROOF. Directly substituting $\varphi(z) = c_1 z^{\lambda_1}$ and $\psi(z) = c_2 z^{\lambda_2}$ into the system (18) we get

$$\begin{cases} c_1 \frac{\Gamma(1+\lambda_1)}{\Gamma(1+\lambda_1-\alpha)} z^{\lambda_1-\alpha} = c_2 (a_1 z^{\lambda_2} + a_2 \lambda_2 z^{\lambda_2}), \\ c_2 \frac{\Gamma(1+\lambda_2)}{\Gamma(1+\lambda_2-\alpha)} z^{\lambda_2-\alpha} = c_1^{2m+1} z^{2\lambda_1 m} (b_1 z^{\lambda_1} + b_2 \lambda_1 z^{\lambda_1}). \end{cases} \quad (19)$$

The powers of z should be equal in both equations of the above system, so we get

$$\lambda_1 = -\frac{\alpha}{m}, \quad \lambda_2 = -\frac{(m+1)\alpha}{m}.$$

By the assumption of the lemma, we can see that $\lambda_1 > -1$ and $\lambda_2 > -1$. Considering the obtained λ_1 and λ_2 , we get c_1 and c_2 as in the lemma. \square

The reduced system corresponding to U_4 of Case 2.1 satisfies the conditions of Lemma 2, which means we have a solution $(\varphi(z), \psi(z))$ as in the lemma. From Table 6 and Table 10 we get the following explicit invariant solution to the system (1) with the condition of $m < 0$ or $m > \frac{\alpha}{1-\alpha}$:

$$\begin{cases} u(x, t) = \left[\frac{m^2}{k^2(m+1)(2m+1)} \frac{\Gamma(-\frac{\alpha}{m})}{\Gamma(-\frac{(2m+1)\alpha}{m})} \right]^{\frac{1}{2m}} x^{\frac{1}{m}} t^{-\frac{\alpha}{m}}, \\ v(x, t) = \left[\frac{m^{2m+2}}{k^2(m+1)^{4m+1}(2m+1)} \frac{\Gamma(-\frac{\alpha}{m})^{2m+1}}{\Gamma(-\frac{(m+1)\alpha}{m})^{2m} \Gamma(-\frac{2m+1}{m}\alpha)} \right]^{\frac{1}{2m}} x^{\frac{m+1}{m}} t^{-\frac{(m+1)\alpha}{m}}. \end{cases} \quad (20)$$

Even though the reduced systems corresponding to U_2 of Case 2.1 and U_3 of Case 2.2 are of the form (18) these systems do not satisfy the conditions of the Lemma 2.

5.3. Reduced system of ODE_5 of Case 2.1

By direct integration of the reduced system we get the following implicit solution with the condition $m < 0$ or $m > \frac{\alpha}{1-\alpha}$:

$$\begin{cases} x = \left(\frac{k^2}{(m+1)} \frac{\Gamma(1 - \frac{(m+1)\alpha}{m})^{2m} \Gamma(1 - \frac{(2m+1)\alpha}{m})}{\Gamma(1 - \frac{\alpha}{m})^{2m+1}} \right)^{\frac{1}{2m+2}} \int_{\psi_0}^{\psi} \frac{d\theta}{(\theta^2 + c_1)^{\frac{1}{2m+2}}} + c_2, \\ \varphi = \left(\frac{m+1}{k^2} \frac{\Gamma(1 - \frac{(m+1)\alpha}{m})^2}{\Gamma(1 - \frac{\alpha}{m}) \Gamma(1 - \frac{(2m+1)\alpha}{m})} (\psi^2 + c_1) \right)^{\frac{1}{2m+2}}, \end{cases} \quad (21)$$

where ψ_0 is appropriately chosen lower bound and c_1, c_2 are constants. The invariant solution $(u(x, t), v(x, t))$ can be obtained via the above implicit solution and Table 6. Integrating the above solution explicitly is

difficult, but for some particular values of the parameters c_1 , c_2 and m , we may find explicit invariant solutions. For example, for $c_1 = 0$ case we obtain the following invariant solution

$$\begin{cases} u(x, t) = \left(\frac{m^2}{k^2(m+1)(2m+1)} \frac{\Gamma(-\frac{\alpha}{m})}{\Gamma(-\frac{(2m+1)\alpha}{m})} \right)^{\frac{1}{2m}} (x - c_2)^{\frac{1}{m}} t^{-\frac{\alpha}{m}}, \\ v(x, t) = \left(\frac{m^{2m+2}}{k^2(m+1)^{4m+1}(2m+1)} \frac{1}{\Gamma(-\frac{(m+1)\alpha}{m})^{2m}} \frac{\Gamma(-\frac{\alpha}{m})^{2m+1}}{\Gamma(-\frac{(2m+1)\alpha}{m})} \right)^{\frac{1}{2m}} (x - c_2)^{\frac{m+1}{m}} t^{-\frac{(m+1)\alpha}{m}}, \end{cases} \quad (22)$$

which corresponds exactly to the one obtained in [17] for $m = \frac{1}{2}$ and $k = 1$. Observe that (20) is invariant under the symmetries X_2 and X_4 , which means (22) is an invariant solution of not only ODE_5 but also of ODE_4 when $c_2 = 0$. Substituting $m = -\frac{1}{2}$ and $c_1 \neq 0$ into (21) we get another explicit invariant solution

$$\begin{cases} u(x, t) = \frac{c_1}{2k^2} \frac{\Gamma(1+\alpha)^2}{\Gamma(1+2\alpha)} t^{2\alpha} \left[\tan^2 \left(\frac{\sqrt{c_1}}{2k^2} \Gamma(1+\alpha)(x - c_2) \right) + 1 \right], \\ v(x, t) = \sqrt{c_1} t^\alpha \tan \left(\frac{\sqrt{c_1}}{2k^2} \Gamma(1+\alpha)(x - c_2) \right). \end{cases} \quad (23)$$

5.4. Reduced system of ODE_2 of Case 2.2

One may easily check that the reduced system has the following solution

$$\begin{cases} \varphi(z) = a_1 \frac{\Gamma(\alpha)}{\Gamma(2\alpha)} z^{2\alpha-1} \\ \psi(z) = cz^{\alpha-1}. \end{cases}$$

Then we obtain the following invariant solution of the system (1) using Table 10

$$\begin{cases} u(x, t) = a_1 \frac{\Gamma(\alpha)}{\Gamma(2\alpha)} t^{2\alpha-1} \\ v(x, t) = a_2 c t^{\alpha-1} + a_1 x t^{\alpha-1}, \end{cases} \quad (24)$$

where $(a_1, a_2) \in \{(a, \pm 1) | a \in \mathbb{R}\}$ and c is a constant.

5.5. Reduced system of ODE_4 of Case 2.2

The solution to the corresponding reduced system is

$$\begin{cases} \varphi(x) = \left(-\frac{a\Gamma(\alpha+1)}{k^2(1-2\alpha)} x + c_1 \right)^{1-2\alpha} \\ \psi(x) = \frac{k^2(2\alpha-1)}{2a(1-\alpha)} \frac{\Gamma(2\alpha)}{\Gamma(\alpha)\Gamma(\alpha+1)} \left(\frac{a\Gamma(\alpha+1)}{k^2(2\alpha-1)} x + c_1 \right)^{2(1-\alpha)} + c_2, \end{cases}$$

which is then used to find the following explicit invariant solution via Table 10

$$\begin{cases} u(x, t) = \left(-\frac{a\Gamma(\alpha+1)}{k^2(1-2\alpha)} x + c_1 \right)^{1-2\alpha} t^{2\alpha-1} \\ v(x, t) = \left(\frac{k^2(2\alpha-1)}{2a(1-\alpha)} \frac{\Gamma(2\alpha)}{\Gamma(\alpha)\Gamma(\alpha+1)} \left(\frac{a\Gamma(\alpha+1)}{k^2(2\alpha-1)} x + c_1 \right)^{2(1-\alpha)} - \alpha a \ln(t) + c_2 \right) t^{\alpha-1}. \end{cases} \quad (25)$$

We have shown the group invariant solutions except the cases corresponding to U_2 , U_3 of Case 2.1 and U_3 of Case 2.2.

6. Conclusion

In this work, we studied a class of nonlinear evolution system of time fractional partial differential equations applying the Lie symmetry analysis. We gave a complete group classification of the system depending on an arbitrary function $b(u)$. Also we give one-dimensional optimal systems of Lie algebras of infinitesimal symmetries and reduced the fractional partial differential equation system into fractional and non-fractional systems of ordinary differential equations corresponding to the symmetries of the optimal systems for the case of $0 < \alpha < 1$. Furthermore the exact invariant solutions to the studied system were given with the help of the reduced systems.

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